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A brief introduction to the Bethe ansatz in $\mathcal{N} = 4$ super-Yang–Mills

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Abstract

We give a brief introduction to the application of the Bethe ansatz to the study of planar $\mathcal{N} = 4$ super-Yang–Mills. The emphasis is on one-loop integrability in the $SU(2)$ sector. We use the Bethe ansatz to find the anomalous dimensions for certain operators and compare these results with string theory predictions based on the AdS/CFT correspondence.

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1. Introduction

Over the last several years, there has been a large amount of work exploiting the apparent integrability of planar $\mathcal{N} = 4$ super-Yang–Mills. Integrability is useful because it allows us to compute quantities which would normally be impossible to compute. The main tool is the Bethe ansatz, first proposed in 1931 as a means of solving the one-dimensional ferromagnet. Today this system is sometimes known as the Heisenberg XXX model. This is probably the simplest integrable model, but nevertheless, it appears in $\mathcal{N} = 4$ super-Yang–Mills [1]!

At the end of 1997, Maldacena proposed that the gauge theory $\mathcal{N} = 4$ super-Yang–Mills is dual to a string theory on a curved background [2]. Several months later, Gubser, Klebanov and Polyakov [3], and intently Witten [4] made this correspondence more precise. The problem with verifying Maldacena's conjecture is that the string theory is best understood when the gauge coupling is large, but perturbative super-Yang–Mills is best understood when the coupling is small. So much of the work in the past several years has been trying to improve techniques so that the string theory and the perturbative gauge theory can be directly compared.

It is believed that integrability will play a major role in this endeavour. The suggestion that the system is integrable implies that it is solvable. Presumably, if perturbative Yang–Mills is integrable, we will no longer be limited to small couplings. For the string theory, we know

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that it is classically integrable. But it is not known if the integrability holds up once quantum corrections are taken into account, although there has been some recent evidence that this indeed will be the case. The size of the quantum corrections is determined by the inverse gauge coupling, so if we can establish integrability for the quantum string, then there is hope that we can take the coupling down to smaller values and still have control of the system.

These notes are a brief introduction to this fascinating subject. They roughly follow a series of lectures given at the ICTP spring school on string theory in 2004. The notes are intended for those just starting out in the subject, or for those involved in the more formal aspects of integrability and who would like to have a better idea of what has been going on in this particular area. Instead of giving a detailed description of the most general type of operators in the gauge theory, I will mainly consider only those operators that are made up of scalar fields. In fact often I will only consider what is known as the $SU(2)$ sector, and even in this sector, I will only consider the simplest of solutions. However, even this relatively simple set of operators demonstrates a rich structure. I will also mainly concentrate on the integrability of the gauge theory to one-loop order in perturbation theory. It has been now well established that the complete $\mathcal{N} = 4$ theory is integrable at the one-loop level [1, 5]. There is also a fair amount of evidence that the integrability will hold up at higher orders in perturbation theory [6–8], but we will not discuss this in these notes.

This paper has been arranged as follows. In section 2 we discuss the superconformal algebra and the importance of chiral primary operators. In section 3 we construct operators in the gauge theory. In section 4 we compute the one-loop anomalous dimensions for operators made up of scalar fields. In section 5 we relate the computation in section 4 to a one-dimensional spin chain, the Heisenberg XXX model. We also show how to construct the Bethe equations for this system and we describe some solutions to the equations. In section 6 we discuss a corresponding string solution and relate it to results in section 5. In section 7 we give a brief description of references for further reading.

2. The superconformal algebra

$\mathcal{N} = 4$ super-Yang–Mills has a large symmetry group that greatly constrains many of the properties of the theory. This symmetry group is known as the $\mathcal{N} = 4$ superconformal group and we will see it, or one of its subgroups, appearing throughout these lectures.

As the name implies, $\mathcal{N} = 4$ SYM is invariant under conformal transformations. This means that this theory does not have a mass scale. Hence all fields are massless and the coupling constant cannot run, otherwise there would be a scale at which the coupling is, say, of over order 1. The theory is also not confining, since confinement also implies that there exists a scale where the theory stops confining. This is what happens for QCD, where one refers to the QCD scale which is of order 300 MeV and is responsible for the proton mass.

In four dimensions, the conformal algebra has 15 generators. These are the four generators of space-time translations, P_μ, the six generators of Lorentz transformations, $M_{\mu\nu}$, the four special conformal transformations, K_μ and finally the generator of scalings, also known as dilatations, D. These generators satisfy the conformal algebra

$$\begin{aligned}
&[D, P_\mu] = -\mathrm{i}P_\mu \qquad [D, M_{\mu\nu}] = 0 \qquad [D, K_\mu] = +\mathrm{i}K_\mu \\
&[M_{\mu\nu}, P_\lambda] = -\mathrm{i}(\eta_{\mu\lambda}P_\nu - \eta_{\lambda\nu}P_\mu) \qquad [M_{\mu\nu}, K_\lambda] = -\mathrm{i}(\eta_{\mu\lambda}K_\nu - \eta_{\lambda\nu}K_\mu) \\
&[P_\mu, K_\nu] = 2\mathrm{i}(M_{\mu\nu} - \eta_{\mu\nu}D).
\end{aligned} \tag{2.1}$$

Let us now consider a local operator in the field theory $\mathcal{O}(x)$. Under a scaling $x \to \lambda x$, $\mathcal{O}(x)$ scales as $\mathcal{O}(x) \to \lambda^{-\Delta}\mathcal{O}(\lambda x)$, where Δ is the dimension of $\mathcal{O}(x)$. D is the generator of these scalings and its action on $\mathcal{O}(x)$ is

$$[D, \mathcal{O}(x)] = \mathrm{i}\left(-\Delta + x\frac{\partial}{\partial x}\right)\mathcal{O}(x). \tag{2.2}$$

So let us now consider the action of D on the commutator $[K_\mu, \mathcal{O}(0)]$. Using the Jacobi identity we find

$$\begin{aligned}[D, [K_\mu, \mathcal{O}(0)]] &= [[D, K_\mu], \mathcal{O}(0)] + [K_\mu, [D, \mathcal{O}(0)] \\ &= \mathrm{i}[K_\mu, \mathcal{O}(0)] - \mathrm{i}\Delta[K_\mu, \mathcal{O}(0)].\end{aligned} \tag{2.3}$$

Hence, we see that K_μ lowers the dimension by 1. Unitary quantum field theories must have local operators whose dimensions are positive (in fact a nonconstant operator must have $\Delta \geqslant 2$.) Therefore, as we keep lowering the dimension by acting with K_μ, we must eventually reach an operator where

$$[K_\mu, \mathcal{O}(0)] = 0. \tag{2.4}$$

Such an operator is known as a *primary* operator. We can obtain new operators by acting with the conformal algebra on the primaries. These operators are known as *descendants*.

$\mathcal{N} = 4$ SYM is also invariant under supersymmetry transformations. In fact there are eight separate supercharges, along with their conjugates, that generate the supersymmetry transformations, and these combine with the conformal algebra to make a *graded* algebra known as the *superconformal* algebra. In addition to the commutation relations in (2.1) we have the commutation and anti-commutation relations:

$$\begin{aligned}
&\left\{Q^a_\alpha, \widetilde{Q}^{\bar{b}}_{\dot{\alpha}}\right\} = \gamma^\mu_{\alpha\dot{\alpha}}\delta^{a\bar{b}}P_\mu &&\left\{Q^a_\alpha, Q^b_\beta\right\} = \left\{\widetilde{Q}^{\bar{a}}_{\dot{\alpha}}, \widetilde{Q}^{\bar{b}}_{\dot{\beta}}\right\} = 0\\
&\left[P_\mu, Q^a_\alpha\right] = \left[P_\mu, \widetilde{Q}^{\bar{a}}_{\dot{\alpha}}\right] = 0 &&\\
&\left[D, Q^a_\alpha\right] = -\frac{\mathrm{i}}{2}Q^a_\alpha &&\left[D, \widetilde{Q}^{\bar{a}}_{\dot{\alpha}}\right] = -\frac{\mathrm{i}}{2}\widetilde{Q}^{\bar{a}}_{\dot{\alpha}}\\
&\left[M^{\mu\nu}, Q^a_\alpha\right] = \mathrm{i}\sigma^{\mu\nu}_{\alpha\beta}\epsilon^{\beta\gamma}Q^a_\gamma &&\left[M^{\mu\nu}, \widetilde{Q}^{\bar{a}}_{\dot{\alpha}}\right] = \mathrm{i}\sigma^{\mu\nu}_{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{\beta}\dot{\gamma}}\widetilde{Q}^{\bar{a}}_{\dot{\gamma}}.
\end{aligned} \tag{2.5}$$

The indices $\alpha = 1, 2$ $\dot{\alpha} = 1, 2$ are indices for the fundamental representations of the two independent $SU(2)$ algebras that make up the four-dimensional Lorentz group. The indices $a = 1..4$ and $\bar{a} = 1..4$ are indices for the fundamental and anti-fundamental representations of an internal $SU(4) \simeq SO(6)$ symmetry. This symmetry is known as an R-symmetry. We can see that Q^a_α and $\widetilde{Q}^{\bar{a}}_{\dot{\alpha}}$ have dimension 1/2.

The commutator of K_μ with Q^a_α and $\widetilde{Q}^{\bar{a}}_{\dot{\alpha}}$ will give us two sets of fermionic operators with dimension $-1/2$,

$$\left[K^\mu, Q^a_\alpha\right] = \gamma^\mu_{\alpha\dot{\alpha}}\epsilon^{\dot{\alpha}\dot{\beta}}\widetilde{S}^a_{\dot{\beta}} \qquad \left[K^\mu, \widetilde{Q}^{\bar{a}}_{\dot{\alpha}}\right] = \gamma^\mu_{\alpha\dot{\alpha}}\epsilon^{\alpha\beta}S^{\bar{a}}_\beta. \tag{2.6}$$

The operators $S^{\bar{a}}_\alpha$ and $\widetilde{S}^a_{\dot{\alpha}}$ are known as the special conformal supercharges and with the other supercharges gives 32 supercharges in all. The $S^{\bar{a}}_\alpha$ and $\widetilde{S}^a_{\dot{\alpha}}$ also have nontrivial anti-commutation relations,

$$\left\{S^{\bar{a}}_\alpha, \widetilde{S}^b_{\dot{\alpha}}\right\} = \gamma^\mu_{\alpha\dot{\alpha}}\delta^{\bar{a}b}K_\mu \qquad \left\{S^{\bar{a}}_\alpha, S^{\bar{b}}_\alpha\right\} = \left\{\widetilde{S}^a_{\dot{\alpha}}, \widetilde{S}^b_{\dot{\alpha}}\right\} = 0. \tag{2.7}$$

There are also nonzero commutation relations between Q^a_α and $S^{\bar{a}}_\alpha$, which are given by

$$\begin{aligned}
\left\{Q^a_\alpha, S^{\bar{b}}_\beta\right\} &= -\mathrm{i}\varepsilon_{\alpha\beta}\sigma^{ij}_{a\bar{b}}R_{ij} + \sigma^{\mu\nu}_{\alpha\beta}\delta^{a\bar{b}}M_{\mu\nu} - \varepsilon_{\alpha\beta}\delta^{a\bar{b}}D\\
\left\{\widetilde{Q}^{\bar{a}}_{\dot{\alpha}}, \widetilde{S}^b_{\dot{\beta}}\right\} &= +\mathrm{i}\varepsilon_{\dot{\alpha}\dot{\beta}}\sigma^{ij}_{\bar{a}b}R_{ij} + \sigma^{\mu\nu}_{\dot{\alpha}\dot{\beta}}\delta^{\bar{a}b}M_{\mu\nu} - \varepsilon_{\dot{\alpha}\dot{\beta}}\delta^{\bar{a}b}D.
\end{aligned} \tag{2.8}$$

We see that we have a new set of generators in the algebra R_{ij}, where $i, j = 1, \ldots, 6$. These are the generators for the $SU(4)$ R-symmetry. The supercharges are spinors under this symmetry; all other generators are singlets and so commute with R_{ij}.

The supercharges Q^a_α and $\widetilde{Q}^{\bar{a}}_{\dot{\alpha}}$ can be used to generate new operators. The operators generated this way, along with the original operator, make up a *supermultiplet*. Since the supercharges are fermionic, the maximal number of operators in a supermultiplet is $2^8 = 256$. If we can find operators annihilated by some of these supercharges, then the number of operators in the supermultiplet will be reduced.

We now look for operators that will be annihilated by a maximal number of supercharges. It is clear that an operator can be annihilated by all of the special superconformal charges only if the operator is primary, otherwise the anti-commutation relation in (2.7) would be inconsistent. Hence we have

$$\left[S^{\bar{a}}_\alpha, \mathcal{O}(0)\right] = \left[\widetilde{S}^a_{\dot{\alpha}}, \mathcal{O}(0)\right] = 0 \qquad \text{for all} \quad \alpha, \dot{\alpha}, \bar{a}, a. \tag{2.9}$$

We now put the further restriction that

$$\left[Q^a_\alpha, \mathcal{O}(0)\right] = 0 \qquad \text{for some} \quad \alpha, a. \tag{2.10}$$

We call such operators as chiral primaries. Consistency now requires that

$$\begin{aligned}\left[\left\{Q^a_\alpha, S^{\bar{b}}_\beta\right\}, \mathcal{O}(0)\right] &= \left\{Q^a_\alpha, \left[S^{\bar{b}}_\beta, \mathcal{O}(0)\right]\right\} + \left\{S^{\bar{b}}_\beta, \left[Q^a_\alpha, \mathcal{O}(0)\right]\right\} = 0 \\ &= \left[-\mathrm{i}\varepsilon_{\alpha\beta}\sigma^{IJ}_{a\bar{b}} R_{IJ} - \varepsilon\alpha\beta\delta^{a\bar{b}} D + \sigma^{\mu\nu}_{\alpha\beta}\delta^{a\bar{b}} M_{\mu\nu}, \mathcal{O}(0)\right].\end{aligned} \tag{2.11}$$

If $\mathcal{O}(0)$ is a scalar field, $M_{\mu\nu}$ commutes with it and we get a direct relation between the R-charge of $\mathcal{O}(0)$ and its dimension Δ.

To find what the relations are, we first observe that $SO(6)$ is a rank 3 group and so it has three commuting generators, which we choose to be R_{12}, R_{34} and R_{56}. We can write the corresponding charges for these as (J_1, J_2, J_3). The $\sigma^{IJ}_{a\bar{b}}$ are the $SO(6)$ generators in the fundamental representation, and in particular, the commuting generators can be expressed as

$$\sigma^{12}_{a\bar{b}} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix} \quad \sigma^{34}_{a\bar{b}} = \begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix} \quad \sigma^{56}_{a\bar{b}} = \begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}. \tag{2.12}$$

Hence, a primary operator with R-charges $(J_1, 0, 0)$ can be consistently annihilated by Q^1_α, Q^2_α if $\Delta = J_1$. Inspection of (2.8) shows that they are also annihilated by $\widetilde{Q}^3_{\dot{\alpha}}$ and $\widetilde{Q}^4_{\dot{\alpha}}$. Such operators are known as *chiral primaries*, or BPS operators. We could also find chiral primaries with charges $(0, J_2, 0)$ or $(0, 0, J_3)$. Furthermore, if we act on any of these operators with an R^{IJ} we will get an operator that is also a chiral primary.

The great advantage of chiral primaries is that their dimensions are 'protected'. In general, the dimension of an operator depends on the coupling constant of the Yang–Mills theory. However, we have just seen that a chiral primary is annihilated by half the supercharges and so the number of operators in the supermultiplet is $2^4 = 16$. This number cannot continuously change as we vary the coupling, nor can the R-charge change, since it too is an integer. Therefore it must always be a chiral primary no matter what the coupling and so Δ is always equal to the R-charge.

3. Gauge invariant operators in $\mathcal{N} = 4$ SYM

In this section we discuss the gauge invariant operators present in $\mathcal{N} = 4$ super-Yang–Mills with $SU(N)$ gauge group.

In $\mathcal{N} = 4$ SYM, the gauge invariant operators are made up of traces of the various fields that belong to the $\mathcal{N} = 4$ multiplet. The fields themselves all lie in the adjoint representation, which means that under a gauge transformation, the field χ, which is a linear combination of $SU(N)$ generators, transforms as

$$\chi \to \chi + [\varepsilon, \chi] \tag{3.1}$$

where ε is the generator of gauge transformations. A field that transforms like this is said to transform covariantly. Note that χ and ε have an implied spatial dependence. There is also a gauge connection $\mathcal{A}_\mu$ which transforms as

$$\mathcal{A}_\mu \to \mathcal{A}_\mu + \partial_\mu\varepsilon + [\varepsilon, \mathcal{A}_\mu], \tag{3.2}$$

and a covariant derivative

$$\mathcal{D}_\mu \equiv \partial_\mu - [\mathcal{A}_\mu, \cdot] \tag{3.3}$$

where the dot refers to the field being acted on by the covariant derivative. It then follows that $\mathcal{D}_\mu\chi$ also transforms covariantly.

We can then make gauge invariant operators by taking traces of combinations of the various fields. So for example, consider the single trace operator $\mathcal{O}$ given by

$$\mathrm{Tr}[\chi_1\chi_2\ldots\chi_L] \tag{3.4}$$

where χ_i refers to one of the fields in an $\mathcal{N} = 4$ multiplet, with or without covariant derivatives. The operator is local, which means that the fields are evaluated at the same point in space time. Then it is clear that $\mathcal{O}$ will be unchanged under the gauge transformation in (3.1). Clearly, products of traces will also be gauge invariant.

Let us consider the particular fields in $\mathcal{N} = 4$ SYM. First, we have six adjoint scalar fields which are in the vector multiplet of the $SO(6)R$-symmetry. It is convenient to express these as three complex fields, Z, WX, along with their conjugates. If we classify our fields by a 6-tuplet of charges, $(h, L_1, L_2; J_1, J_2, J_3)$, where h is the bare dimension, L_1 and L_2 are the two charges of the Lorentz group $SO(1, 3)$ and J_1, J_2 and J_3 are the three commuting $SO(6)$ R-charges, then the charges for Z, W and X are given by $(1, 0, 0; 1, 0, 0)$, $(1, 0, 0; 0, 1, 0)$ and $(1, 0, 0; 0, 0, 1)$, respectively, with the Lorentz charges being zero since the fields are scalars and $h = 1$, since the bare dimension of a scalar in four dimensions is 1. The conjugate scalars have R-charges with opposite signs.

Now let us consider the operator $\Psi_L \equiv \mathrm{Tr}[Z^L]$. The charges of Ψ_L are $(L, 0, 0; L, 0, 0)$, but based on the arguments of the previous section, this operator satisfies the BPS condition. Hence, Ψ_L is a chiral primary. We will use Ψ_L as the starting point for all other operators. For instance, we can act on Ψ_L with the R-symmetry group. Since Ψ_L is an element of the L-fold symmetric representation of $SO(6)$, then the resulting operator will also be in a symmetric representation. The resulting operators are still BPS. In fact, if we change one of the Z fields to be any other scalar field, say ϕ, then the resulting operator $\mathrm{Tr}[\phi Z^{L-1}]$ is *still* BPS since it is automatically symmetric. So we learn that to make a non-BPS operator we will need at least two of the fields not to be a Z.

There are also 16 fermionic fields that can be included inside the trace, ψ^a_α and $\bar{\psi}^{\bar{a}}_{\dot{\alpha}}$, where the two sets of fermions are spinors under the two different $SU(2)$'s that make up $SO(1, 3)$. The charges for these fields are of the form $(3/2, 0, \pm 1/2; \pm 1/2 \pm 1/2 \pm 1/2)$ and $(3/2, \pm 1/2, 0; \pm 1/2 \pm 1/2 \pm 1/2)$ where the number of $SO(6)$ negative signs is even for the first set and odd for the second.

We may also act on the scalar and spinor fields with covariant derivatives. The covariant derivatives have charges $(1, \pm 1/2, \pm 1/2; 0, 0, 0)$. If we consider $\mathrm{Tr}[Z^{L-1}\mathcal{D}_\mu Z]$, then we have

$$\mathrm{Tr}[Z^{L-1}\mathcal{D}_\mu Z] = \mathrm{Tr}[Z^{L-1}\partial_\mu Z] = \frac{1}{L}\partial_\mu\Psi_L. \tag{3.5}$$

In other words, this operator is a descendant of Ψ_L. We can also consider operators containing the field strength $\mathcal{F}_{\mu,\nu} = [\mathcal{D}_\mu, \mathcal{D}_\nu]$.

It turns out that many operators are related to other operators, through the equations of motion or Bianchi identities. For example, for scalar fields the equations of motion are qualitatively of the form

$$\mathcal{D}^\mu \mathcal{D}_\mu Z = \cdots \tag{3.6}$$

where the right-hand side contains nonderivative terms. Hence, two contracted derivatives acting on a scalar field can always be replaced by nonderivative terms. Since we also have that anti-symmetrized derivatives give the field strength, we can always restrict ourselves to symmetrized derivatives, without contractions.

It is very convenient to construct the fields using bosonic and fermionic creation operators. We first note that an $SO(1,3)$ vector is a bispinor of the two $SU(2)$'s that make up $SO(1,3)$, so the covariant derivative can also be written as $\mathcal{D}_{\alpha\dot{\beta}}$. The field strength is in the $(3,1)+(1,3)$ representation and so we can split it up into two types, the selfdual $\mathcal{F}_{\alpha\beta}$ and the anti-selfdual $\bar{\mathcal{F}}_{\dot{\alpha}\dot{\beta}}$ components, where the indices are symmetrized. We also note that the scalar fields, which are vectors in the $SO(6)R$-symmetry group, is equivalent to the anti-symmetric representation of $SU(4)$, and so can be written as ϕ_{ab} with the indices anti-symmetrized. Also, since the anti-fundamental of $SU(4)$ is the result of anti-symmetrizing three fundamentals, we see that all $SU(4)$ indices are anti-symmetrized and all indices in either of the $SU(2)$'s are symmetrized (since we are assuming that all covariant derivatives will be symmetrized). Hence, we can consider two sets of bosonic creation operators $A^\dagger_\alpha$, $B^\dagger_{\dot{\alpha}}$, one for each $SU(2)$ and a set of fermionic creation operators $C^\dagger_a$, and build the fields as follows: start with the field '1', which we will signify as a state $|0\rangle$. Then the corresponding states for the other fields are

$$\begin{array}{lll}
\mathcal{D}^k\mathcal{F} & \Rightarrow & (A^\dagger)^{k+2}(B^\dagger)^k|0\rangle \\
\mathcal{D}^k\psi_{\alpha,a} & \Rightarrow & (A^\dagger)^{k+1}(B^\dagger)^k C^\dagger_a|0\rangle \\
\mathcal{D}^k\phi_{ab} & \Rightarrow & (A^\dagger)^{k}(B^\dagger)^k C^\dagger_a C^\dagger_b|0\rangle \\
\mathcal{D}^k\bar{\psi}_{\dot{\alpha},\bar{a}} & \Rightarrow & (A^\dagger)^{k}(B^\dagger)^{k+1} C^\dagger_a C^\dagger_b C^\dagger_c|0\rangle \\
\mathcal{D}^k\bar{\mathcal{F}} & \Rightarrow & (A^\dagger)^{k}(B^\dagger)^{k+2} C^\dagger_a C^\dagger_b C^\dagger_c C^\dagger_d|0\rangle.
\end{array} \tag{3.7}$$

Putting these types of fields together in a trace, we can then add indices to the creation operators for each position in the trace and generalize them to $A^\dagger_{\alpha,\ell}$, $B^\dagger_{\dot{\alpha},\ell}$ and $C^\dagger_{a,\ell}$, where $1 \leqslant \ell \leqslant L$ is the position. The operator made up of these fields can then be mapped to a particular state using these creation operators.

If we now take the limit where the number of colours N approaches ∞, then the computation of anomalous dimensions simplifies tremendously. We will explain this more fully in the next section, but in taking this limit we may restrict ourselves to operators composed of single traces only. The dimensions of multitrace operators will just be the sums of the dimensions of the single traces that make up the operator.

Later on in this paper we will also make the further simplifying assumption that the operators are in what is known as the '$SU(2)$' sector. Such operators are made up only of traces of two types of scalar fields, Z and W, without their conjugate fields. The reason that we can restrict the operators to the $SU(2)$ sector is that it is closed to all orders in perturbation theory under operator mixing. This is not too hard to show, just by using the conservation of the R-charges and the Lorentz charges, as well as by making use of the property that operators of different bare dimensions cannot mix in a conformal field theory[2]. By looking at the

[2] A nice explanation of this property can be found in [9].

charges for the other fields, it is straightforward to show that the only single trace operators with the same charges as an operator made up of Z and W fields only, are operators with the same number of Z and W fields as this original operator and with no other fields included. The Z field can be considered the top component of a fundamental $SU(2)$ representation and the W is the bottom component. Note that if we were to add a third field X with charges $(1, 0, 0; 0, 0, 1)$, then the combination ZWX would have the same charges as two fermions and so operators made of the three scalars would not be a closed sector. However, if we limit the operators to include the three scalars plus the two extra fermions, then this *would* be a closed sector, with the fields transforming under the supergroup $SU(2|3)$ [7]. In any case, the $SU(2)$ sector is nontrivial enough to demonstrate much of the integrability found in $\mathcal{N} = 4$ SYM.

Another sector we will consider is the '$SO(6)$' sector. These are operators that are made up of all six types of scalar fields. This sector is not closed in perturbation theory, although it is closed to one-loop order.

4. One-loop anomalous dimensions

In this section we compute the one-loop anomalous dimensions for operators in the $SU(2)$ sector. To find the anomalous dimension of an operator, one considers the two-point correlator of the operator with itself. In particular, one finds that

$$\langle \mathcal{O}(x)\bar{\mathcal{O}}(y)\rangle \approx \frac{1}{|x-y|^{2\Delta}}, \tag{4.1}$$

where the dimension $\Delta = \Delta_0 + \gamma$, with Δ_0 being the bare dimension and γ being the anomalous dimension. The bare dimension is the dimension of the operator if the gauge theory coupling constant is 0. Therefore, for operators made up only of scalar fields, the bare dimension is L, the number of scalar fields inside the trace.

If the gauge coupling is small, then $\gamma \ll \Delta_0$, in which case we can approximate the correlator in (4.1) as

$$\langle \mathcal{O}(x)\bar{\mathcal{O}}(y)\rangle \approx \frac{1}{|x-y|^{2\Delta_0}}(1 - \gamma \ln \Lambda^2 |x-y|^2), \tag{4.2}$$

where Λ is an energy scale. The leading contribution to this correlator is called the tree-level contribution.

Let us now investigate the role of large N. For example, let us consider the operator considered in the previous section, Ψ_L, rescaled to

$$\Psi_{L,r} = \frac{1}{\sqrt{L}N^{L/2}} \operatorname{Tr} Z^L = \frac{1}{\sqrt{L}N^{L/2}} Z^A{}_{\bar{B}} Z^B{}_{\bar{C}} \cdots Z^{\cdots}{}_{\bar{A}}, \tag{4.3}$$

where we have explicitly put the colour indices on the adjoint fields. At tree level, the correlator of a Z field and its conjugate is

$$\left\langle Z^A{}_{\bar{B}}(x)\bar{Z}^{\bar{C}}_D(y)\right\rangle \sim \frac{\delta^A{}_D \delta^{\bar{B}}{}_{\bar{C}}}{|x-y|^2}. \tag{4.4}$$

If we now contract this operator with its conjugate, then the leading contribution to the correlator comes from contracting the individual fields in order, as shown in figure 1(*a*) and (*b*). The contribution of all ordered contractions is

$$\langle \Psi_{L,r}(x)\bar{\Psi}_{L,r}(y)\rangle \sim \frac{LN^L}{(\sqrt{L}N^{L/2})^2|x-y|^{2L}} = \frac{1}{|x-y|^{2L}}, \tag{4.5}$$

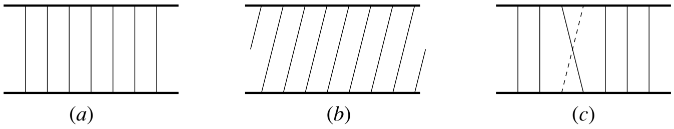

Figure 1. Contractions of fields. The horizontal lines represent the operators and the ordered vertical lines the contractions between the two operators of the individual fields inside the trace. (*a*) and (*b*) are planar while (*c*) is nonplanar.

where we pick up L powers of N from summing over the colours. The contraction shown in figure 1(*c*) is an example of a nonplanar graph, in other words the lines connecting the fields cannot be drawn on the plane. Instead one must lift the line out of the plane to connect to the field in the other operator. If we carefully count the sums over the colours, we see that there are two less factors of N. Hence this diagram is suppressed by a factor of $1/N^2$. Note that this analysis is valid so long as $L \ll N$, otherwise the number of possible contractions will overwhelm the suppression factors.

It is also convenient to write (4.5) in terms of its Fourier transform,

$$\langle \Psi_{L,r}(x)\bar{\Psi}_{L,r}(y)\rangle \sim \int \prod_{k=1}^{L}\left[\exp(-\mathrm{i}p_k\cdot(x-y))\frac{\mathrm{d}^4p_k}{(2\pi)^4}\frac{\mathrm{i}}{p_k^2}\right], \tag{4.6}$$

which will be convenient for corrections to γ.

To find the one-loop anomalous dimension, let us examine the bosonic part of the $\mathcal{N}=4$ action, which is given by

$$S = \frac{1}{2g_{\mathrm{YM}}^2}\int \mathrm{d}^4x\left\{-\frac{1}{2}\operatorname{Tr}\mathcal{F}^2 + \operatorname{Tr}\mathcal{D}_\mu\phi_i\mathcal{D}^\mu\phi^i - \sum_{i<j}\operatorname{Tr}[\phi_i,\phi_j]^2\right\}, \tag{4.7}$$

where g_{YM} is the Yang–Mills coupling. The fields ϕ_i refer to the six bosonic scalars, where we will often call the index i the flavour index. Their relation to Z and W is given by

$$Z = \tfrac{1}{\sqrt{2}}(\phi_1 + \mathrm{i}\phi_2) \qquad W = \tfrac{1}{\sqrt{2}}(\phi_3 + \mathrm{i}\phi_4). \tag{4.8}$$

Now there are several Feynman graphs that contribute to the anomalous dimension of an operator, but remarkably, because of the power of the superconformal algebra, we only need to consider one type of Feynman graph. Happily, it is the easiest graph to compute. If we study (4.7), we see that the last term is a 4-scalar interaction term which we can expand to

$$\frac{1}{g_{\mathrm{YM}}^2}\sum_{i<j}(\operatorname{Tr}\phi_i\phi_i\phi_j\phi_j - \operatorname{Tr}\phi_i\phi_j\phi_i\phi_j). \tag{4.9}$$

This vertex should then be inserted in the correlator and be Wick contracted with two neighbouring fields in the incoming operator and two neighbouring fields in the outgoing operator so that the resulting Feynman graph is planar. In particular, we should consider the correlator

$$\begin{aligned}&\left\langle(\phi_{i_1}\phi_{i_2})^A{}_{\bar{B}}(x)\frac{\mathrm{i}}{g_{\mathrm{YM}}^2}\int \mathrm{d}^4z\sum_{i<j}(\operatorname{Tr}\phi_i\phi_i\phi_j\phi_j - \operatorname{Tr}\phi_i\phi_j\phi_i\phi_j)(\phi_{j_1}\phi_{j_2})^C{}_{\bar{D}}(y)\right\rangle\\ &\quad = N\delta^A{}_{\bar{D}}\delta^C{}_{\bar{B}}\frac{\mathrm{i}g_{\mathrm{YM}}^2N}{2}\int\frac{\mathrm{d}^4p_1}{(2\pi)^4}\frac{\mathrm{d}^4p_2}{(2\pi)^4}\frac{\mathrm{d}^4p_3}{(2\pi)^4}\frac{\mathrm{d}^4p_4}{(2\pi)^4}(2\pi)^4\delta^4(p_3+p_4-p_1-p_2)\\ &\quad\quad\times\frac{\mathrm{i}}{p_1^2}\frac{\mathrm{i}}{p_2^2}\frac{\mathrm{i}}{p_3^2}\frac{\mathrm{i}}{p_4^2}\,\mathrm{e}^{\mathrm{i}(p_3+p_4)y-\mathrm{i}(p_1+p_2)x}\left(4\delta^{j_2}{}_{i_1}\delta^{j_1}{}_{i_2} - 2\delta^{j_1}{}_{i_1}\delta^{j_2}{}_{i_2} - 2\delta_{i_1i_2}\delta^{j_1j_2}\right).\end{aligned} \tag{4.10}$$

The set of delta functions for the flavour indices arise from the two terms in (4.9). There are four ways to contract the indices in (4.9) with the incoming and outgoing fields. The first term in (4.9) always reverses the indices between the incoming and outgoing fields. The last term either contracts the incoming indices with the outgoing indices in order, or it contracts incoming to incoming and outgoing to outgoing. Note that there are two factors of N in (4.10), coming from sums over colour factors, while the correlator $\langle(\phi_{i_1}\phi_{i_2})^A{}_{\bar{B}}(x)(\phi_{j_1}\phi_{j_2})^C{}_{\bar{D}}(y)\rangle$ has only one such factor. In fact, it is not difficult to see that for all planar graphs, every factor of g_{YM}^2 comes with a factor of N. Hence, it is convenient to define a new expansion parameter $\lambda \equiv g_{\mathrm{YM}}^2 N$, also known as the 't Hooft parameter.

Another important feature is that the integrals in (4.10) lead to log divergences and need to be cut off at an ultraviolet scale Λ. It is then not difficult to show that (4.10) is given by

$$\frac{N\delta^A{}_{\bar{D}}\delta^C{}_{\bar{B}}}{|x-y|^2}\frac{\lambda}{16\pi^2}\ln(\Lambda^2|x-y|^2)\left(2\delta^{j_2}{}_{i_1}\delta^{j_1}{}_{i_2}-\delta^{j_1}{}_{i_1}\delta^{j_2}{}_{i_2}-\delta_{i_1i_2}\delta^{j_1j_2}\right). \tag{4.11}$$

There will also be other contributions to the one-loop anomalous dimension. These can come from gluon exchange between scalar fields or self-energy diagrams. We could compute these contributions, but we do not actually need to do this. It is sufficient to note that since gluons have no R-charges and since the flavour of a scalar going into a self-energy diagram is the same coming out, these types of diagrams will not change the flavour indices. Hence, these interactions will only lead to terms where the flavour indices are grouped as

$$\delta^{j_1}{}_{i_1}\delta^{j_2}{}_{i_2} \tag{4.12}$$

between neighbouring fields in the correlators.

Let us now apply these results to $\langle\mathcal{O}_1(x)\mathcal{O}_2(y)\rangle$, where the $\mathcal{O}(x)$ are in the $SO(6)$ sector, that is they are composed of L scalar fields with explicit flavour indices:

$$\mathcal{O}_1(x)=\frac{1}{N^{L/2}}\operatorname{Tr}\left(\phi_{i_1}(x)\cdots\phi_{i_L}(x)\right)\qquad \mathcal{O}_2(x)=\frac{1}{N^{L/2}}\operatorname{Tr}\left(\phi_{j_1}(x)\cdots\phi_{j_L}(x)\right). \tag{4.13}$$

Examining (4.11), we see that the one-loop contribution to the correlator is

$$\langle\mathcal{O}_1(x)\mathcal{O}_2(y)\rangle=\frac{\lambda}{16\pi^2}\frac{\ln\Lambda^2|x-y|^2}{|x-y|^{2L}}\sum_{\ell=1}^{L}(2P_{\ell,\ell+1}-K_{\ell,\ell+1}-C)\delta^{j_1}{}_{i_1}\delta^{j_2}{}_{i_2}\cdots\delta^{j_L}{}_{i_L}+\text{cycles}. \tag{4.14}$$

$P_{\ell,\ell+1}$ is an exchange operator and acts on the flavour indices as

$$P_{\ell,\ell+1}\delta^{j_1}{}_{i_1}\cdots\delta^{j_\ell}{}_{i_\ell}\delta^{j_{\ell+1}}{}_{i_{\ell+1}}\cdots\delta^{j_L}{}_{i_L}=\delta^{j_1}{}_{i_1}\cdots\delta^{j_{\ell+1}}{}_{i_\ell}\delta^{j_\ell}{}_{i_{\ell+1}}\cdots\delta^{j_L}{}_{i_L}. \tag{4.15}$$

$K_{\ell,\ell+1}$ is called a trace operator and its action on the flavour indices is

$$K_{\ell,\ell+1}\delta^{j_1}{}_{i_1}\cdots\delta^{j_\ell}{}_{i_\ell}\delta^{j_{\ell+1}}{}_{i_{\ell+1}}\cdots\delta^{j_L}{}_{i_L}=\delta^{j_1}{}_{i_1}\cdots\delta_{i_\ell,i_{\ell+1}}\delta^{j_\ell,j_{\ell+1}}\cdots\delta^{j_L}{}_{i_L}. \tag{4.16}$$

We also have a constant C, to be determined shortly, which has contributions from gluon exchange, self-energy and the 4-scalar interaction. The sum over cycles refers to contractions where the j indices are uniformly shifted.

If we add this correction to the tree-level correlator, then the correlator up to the one-loop level is

$$\langle\mathcal{O}_1(x)\mathcal{O}_2(y)\rangle=\frac{1}{|x-y|^{2L}}\left(1-\frac{\lambda}{16\pi^2}\ln\Lambda^2|x-y|^2\sum_{\ell=1}^{L}(C-2P_{\ell,\ell+1}+K_{\ell,\ell+1})\right)\\ \times\delta^{j_1}{}_{i_1}\cdots\delta^{j_L}{}_{i_L}+\text{cycles}. \tag{4.17}$$

If we compare this result to (4.2), we see that because of mixing between the operators, the anomalous dimension should be thought of as an operator, Γ, whose eigenvalues are γ. In particular, we find for operators made up of scalar fields, that Γ is given by [1]

$$\Gamma = \frac{\lambda}{16\pi^2}\sum_{\ell=1}^{L}(C - 2P_{\ell,\ell+1} + K_{\ell,\ell+1}). \tag{4.18}$$

Note that Γ is also the one-loop correction to the dilation operator D in the $SO(6)$ sector. To find C, we note that an operator in the symmetric representation of $SO(6)$ is a chiral primary operator, which means its anomalous dimension is zero. The symmetric representations are traceless; hence $P_{\ell,\ell+1}$ and $K_{\ell,\ell+1}$ acting on such operators are 1 and 0, respectively. Hence we see that $C = 2$.

If we now restrict the operators to the $SU(2)$ sector, then there is no contribution from $K_{\ell,\ell+1}$ since the operators only have Z and W and not their conjugate fields. In this case we have

$$\Gamma_{SU(2)} = \frac{\lambda}{8\pi^2}\sum_{\ell=1}^{L}(1 - P_{\ell,\ell+1}). \tag{4.19}$$

Remarkably, this operator is the Hamiltonian of the XXX Heisenberg periodic spin chain with L lattice sites. In fact, we can think of the Z field as an up spin and the W field as a down spin, and the energies of the various states of the spin chain correspond to the anomalous dimensions of the operators. Note that the Hamiltonian can also be written as

$$\Gamma_{SU(2)} = \frac{\lambda}{8\pi^2}\sum_{\ell=1}^{L}\left(\frac{1}{2} - 2\vec{S}_\ell\cdot\vec{S}_{\ell+1}\right). \tag{4.20}$$

Hence the spin chain is ferromagnetic and the ground state has all spins aligned, with total spin $L/2$. But this is the symmetric representation, in other words, this corresponds to the chiral primary operator. So nonchiral primaries can be thought of as excitations about the ground state.

5. Using the Bethe ansatz to compute anomalous dimensions

Now that we have mapped the one-loop $SU(2)$ sector to the XXX Heisenberg model, we can use all of the machinery of integrability to compute the anomalous dimensions in this sector. While we will not prove it here, it also turns out that the $SO(6)$ sector [1] as well as the full $\mathcal{N} = 4$ set of operators will also lead to integrable spin chains [5].

Since the Heisenberg model is integrable, it basically means that the scattering of excitations about the ground state can be reduced to two-body scattering. The simplest excitation is known as a 'magnon'. One starts with the state with all spins up, which is a ground state, and flips one of the spins. Such a state is not an eigenstate of the Hamiltonian, but the following superposition is

$$|p\rangle = \frac{1}{\sqrt{L}}\sum_{\ell=1}^{L} \mathrm{e}^{\mathrm{i}p\ell}|\ell\rangle, \tag{5.1}$$

where $|\ell\rangle$ refers to the state with a down spin at position ℓ. In order for the state to be single valued, we require that $p = 2\pi n/L$. One can also readily see that the energy of this state is

$$E = \epsilon(p_1) = \frac{\lambda}{2\pi^2}\sin^2\frac{p}{2}. \tag{5.2}$$

However, while such a state exists for the spin chain, there is no corresponding operator in the gauge theory. This is because the spin chain wavefunction must be invariant if all sites are translated *completely* around the circle. However, because of the trace, the gauge invariant operators are invariant if all the fields are shifted over by just one site. Hence, the only single magnon solution that would correspond to a gauge invariant operator would have $p = 2\pi n$. But this is just the ground state which is equivalent to no magnons at all. This condition for the gauge invariant operators is known as the 'trace condition'.

We next consider a state with two magnons. As a first attempt at writing down a state, we consider

$$|p_1, p_2\rangle = \frac{1}{L}\sum_{\ell_1,\ell_2} \exp(\mathrm{i}p_1\ell_1 + \mathrm{i}p_2\ell_2)|\ell_1, \ell_2\rangle, \tag{5.3}$$

where ℓ_1 and ℓ_2 are the positions of the two down spins. However, this cannot be completely correct since among other things, the wavefunction is non-zero if $\ell_1 = \ell_2$. But this is impossible since a site has either an up spin or a down spin. It cannot have two down spins. This means that the wavefunction must be zero if $\ell_1 = \ell_2$, which further implies that the magnons interact with each other.

To sort this out, let us take the limit $L \to \infty$ and consider the state

$$|p_1, p_2\rangle = \sum_{\ell_1<\ell_2} \exp(\mathrm{i}p_1\ell_1 + \mathrm{i}p_2\ell_2)|\ell_1, \ell_2\rangle + S(p_1, p_2)\sum_{\ell_1<\ell_2} \exp(\mathrm{i}p_2\ell_1 + \mathrm{i}p_1\ell_2)|\ell_1, \ell_2\rangle. \tag{5.4}$$

This describes the state where a magnon with momentum p_1 comes in from the left to scatter with a magnon coming from the right with momentum p_2. Hence $S(p_1, p_2)$ is the S-matrix. If the magnons are well separated, then they may be treated individually, in which case we find that the energy of this state is just the sum of the magnon energies, $E = \epsilon(p_1) + \epsilon(p_2)$. To find the S-matrix, we observe that the ingoing and outgoing parts of the state will overlap with each other when acted on by the Hamiltonian at $\ell_2 = \ell_1 + 1$. Consistency then leads to the equation

$$\begin{aligned}\frac{\lambda}{8\pi^2}&(-\mathrm{e}^{-\mathrm{i}p_1+\mathrm{i}p_2} - \mathrm{e}^{2\mathrm{i}p_2} - (\mathrm{e}^{\mathrm{i}p_1-\mathrm{i}p_2} + \mathrm{e}^{2\mathrm{i}p_1})S + 2\mathrm{e}^{\mathrm{i}p_2} + 2\mathrm{e}^{\mathrm{i}p_1}S)|\ell, \ell+1\rangle \\ &= \frac{\lambda}{8\pi^2}(4 - \mathrm{e}^{\mathrm{i}p_1} - \mathrm{e}^{-\mathrm{i}p_1} - \mathrm{e}^{\mathrm{i}p_2} - \mathrm{e}^{-\mathrm{i}p_2})(\mathrm{e}^{\mathrm{i}p_2} + \mathrm{e}^{\mathrm{i}p_1}S)|\ell, \ell+1\rangle.\end{aligned} \tag{5.5}$$

Thus, we find

$$S(p_1, p_2) = -\frac{\cos\frac{p_1+p_2}{2} - \exp\left(\mathrm{i}\frac{p_2-p_1}{2}\right)}{\cos\frac{p_1+p_2}{2} - \exp\left(-\mathrm{i}\frac{p_2-p_1}{2}\right)}. \tag{5.6}$$

It is convenient to introduce a new quantity u, called the rapidity, where

$$u = \frac{1}{2}\cot\frac{p}{2}, \tag{5.7}$$

in which case the S-matrix becomes

$$S = \frac{u_2 - u_1 + \mathrm{i}}{u_2 - u_1 - \mathrm{i}}. \tag{5.8}$$

Let us now put L finite again. Since the interactions between the magnons are local, there is no correction to the S-matrix due to finite size effects. Moreover, since the only thing magnons can do when they scatter is exchange their momentum, we can extend the two-magnon case to an arbitrary number of magnons M. To this end, suppose we have M magnons and we want to determine the allowed momenta for the magnons. For a single magnon, we saw that the periodicity condition leads to a quantization condition for the magnon momentum. In other words, when we transport the magnon around the circle, the wavefunction must come back to itself. If we now have M magnons, then when we transport a magnon around the circle, it will pass through the other $M-1$ magnons. Hence, our quantization condition is

$$\mathrm{e}^{\mathrm{i}p_jL}\prod_{k\neq j}^{M} S(p_j,p_k)=1, \tag{5.9}$$

or in terms of the rapidity

$$\left(\frac{u_j+\mathrm{i}/2}{u_j-\mathrm{i}/2}\right)^L=\prod_{k\neq j}^{M}\frac{u_j-u_k+\mathrm{i}}{u_j-u_k-\mathrm{i}}. \tag{5.10}$$

The energy, written in terms of the u_j, is

$$E=\frac{\lambda}{8\pi^2}\sum_{j=1}^{M}\left(\frac{\mathrm{i}}{u_j+\mathrm{i}/2}-\frac{\mathrm{i}}{u_j-\mathrm{i}/2}\right). \tag{5.11}$$

and the total momentum P of all the magnons is given by

$$\mathrm{e}^{\mathrm{i}P}=\prod_{j=1}^{M}\frac{u_j+\mathrm{i}/2}{u_j-\mathrm{i}/2}=1. \tag{5.12}$$

The last equality in (5.12) is due to the trace condition. Note that in (5.8) if $u_j=u_k$ then $S(u_j,u_k)=-1$, in which case the wavefunction in (5.4) is zero. Hence, we must insist that $u_j\neq u_k$ for all $j\neq k$.

The M equations in (5.10) were written down by Bethe in 1931 and are appropriately called the Bethe equations [10]. Hence, to find the one-loop anomalous dimensions in the $SU(2)$ sector it is only necessary to solve these equations!

However, in practice, solving the Bethe equations is in general not possible. In the remaining part of this section I will review a few cases where one can find solutions. The first such solution involves two magnons. In this case, the trace condition in (5.12) leads to $u_2=-u_1$. Hence the Bethe equations reduce to

$$\left(\frac{u_j+\mathrm{i}/2}{u_j-\mathrm{i}/2}\right)^{L-1}=1. \tag{5.13}$$

Hence, we have that $p_j=2\pi n/(L-1)$ for any integer n and the energy of this state is

$$E=\frac{\lambda}{\pi^2}\sin^2\frac{\pi n}{L-1}. \tag{5.14}$$

The first nontrivial case occurs for $L=4$, where by choosing $n=1$ we find[3] $E=\frac{3\lambda}{4\pi^2}$. Note that this corresponds to the operator $\mathrm{Tr}\,Z^2W^2-\mathrm{Tr}\,ZWZW$. It turns out that this operator is a superpartner of the *Konishi operator*, $\mathrm{Tr}\,\phi_i\phi^i$, which is an $SO(6)$ singlet. Since they are superpartners they have the same anomalous dimension. Hence we can find the anomalous dimension for Konishi, an operator outside the $SU(2)$ sector, by considering another operator in the $SU(2)$ sector. This has been used to great effect for higher loop contributions [6]. Note that in the case of large L, the result in (5.14) approaches the result for two noninteracting magnons of opposite momentum [11].

The second example I will consider consists of a large number of magnons, where $L\gg 1$, $M\gg 1$ and $\alpha\equiv M/L$ finite. We will call α the filling fraction. If we assume that the

[3] The case $n=2$ for $L=4$ and $n=1$ for $L=3$ are not allowed, since in these cases the two rapidities would be equal.

momentum of the individual magnons is of order $1/L$, then $p \sim 1/u$ and the Bethe equations can be approximated as

$$e^{iL/u_j} \approx \prod_{k\neq j}^{M} \exp\left(\frac{2i}{u_j - u_k}\right). \tag{5.15}$$

Taking the log on both sides of the equation, we are left with

$$\frac{L}{u_j} - 2\pi n_j \approx 2\sum_{k\neq j}^{M} \frac{1}{u_j - u_k}, \tag{5.16}$$

where n_j is an integer and corresponds to the branch of the log. If we now choose the rescaled variables $x_j = u_j/L$ and assume that the x_j fall into dense sets, then the sums can be replaced by integrals and we obtain the integral equation [12]

$$\frac{1}{x} - 2\pi n_J = 2\sum_K \fint_{\mathcal{C}_K} \frac{dx'\rho(x')}{x - x'} \qquad x \in \mathcal{C}_J, \tag{5.17}$$

where $\rho(x)$ is the density

$$\rho(x) = \frac{1}{L}\sum_{k=1}^{M} \delta(x - x_j). \tag{5.18}$$

The bar through the integral indicates that this is a principle-valued integral and the $\mathcal{C}_J$ are a collection of contours that the rapidities lie on. If two rapidities have the same value for n_j, then they will lie on the same contour.

In this limit, the total momentum and energy are given by

$$P = \sum_K \int_{\mathcal{C}_K} \frac{dx\,\rho(x)}{x} \qquad E = \frac{\lambda}{8\pi^2}\sum_K \int_{\mathcal{C}_K} \frac{dx\,\rho(x)}{x^2}. \tag{5.19}$$

In fact, the XXX chain has L different conserved quantities, which we may call $\mathcal{Q}_n$, and these quantities are given by

$$\mathcal{Q}_n = \frac{1}{L^{n-1}}\sum_K \int_{\mathcal{C}_K} \frac{dx\,\rho(x)}{x^n}. \tag{5.20}$$

If we define the 'resolvent' $G(x)$ as

$$G(x) = \sum_K \int_{\mathcal{C}_K} \frac{dx'\rho(x')}{x - x'}, \tag{5.21}$$

then we see that $G(x)$ is the generator of the conserved charges [13, 14]. In other words, if we Taylor expand about $x = 0$, we find

$$G(x) = \sum_n \frac{x^n}{n!} G^{(n)}(0) = -\sum_n \frac{x^n L^{n-1}}{n!} \mathcal{Q}_n. \tag{5.22}$$

Hence,

$$\mathcal{Q}_n = -\frac{1}{L^{n-1}} G^{(n)}(0). \tag{5.23}$$

The equation in (5.17) is solved using the Riemann–Hilbert method. We will consider the simplified case where all roots lie on a single contour [12, 16, 17] , so the equation becomes

$$\frac{1}{x} - 2\pi n = 2\fint_{\mathcal{C}} \frac{dx'\rho(x')}{x - x'} \qquad x \in \mathcal{C}. \tag{5.24}$$

The goal is to find the density that satisfies the equation, or more precisely, the resolvent that arises from the density. First, let us get an idea about the general shape of the contour. In order for the energy to be real, the contour needs to be symmetric under complex conjugation in the x plane. Second, we note that the integral equation is similar to one that arises when considering large N matrix models [15]. For these, one looks for extrema of the effective action of the matrix eigenvalues. If the matrix potential is $V(x)$, then these extrema satisfy the force balancing equation

$$0 = -V'(x) + 2\int \frac{\mathrm{d}x'\rho(x')}{x - x'}, \tag{5.25}$$

where the second term on the rhs is due to the repulsive force among the eigenvalues. Assuming that the central potential is attractive, the eigenvalues will arrange themselves on the real line, with the attractive central potential balanced by the repulsive potential between the eigenvalues. Applying the same logic to (5.24), we see that the potential is $V(x) = \ln x - 2\pi n x$. Hence, this potential is repulsive, so the rapidities would be driven away from each other if they were constrained to be real. However, the rapidities can be complex, and so what will happen is that they will be pushed into the imaginary direction. Hence, the contour $\mathcal{C}$ will cross the real axis at one point and be symmetric under complex conjugation.

Now, for large x, the resolvent $G(x)$ is approximately

$$G(x) \approx \frac{1}{x}\int_{\mathcal{C}} \mathrm{d}x'\rho(x') = \frac{\alpha}{x}, \qquad x \to \infty. \tag{5.26}$$

We also have from (5.19), (5.22) and the trace condition that

$$G(0) = -2\pi m, \tag{5.27}$$

where m is an integer. Finally, from (5.24) we see that $G(x)$ has a square root branch cut along the contour, where on the contour

$$G(x + \mathrm{i}0) + G(x - \mathrm{i}0) = \frac{1}{x} - 2\pi n, \tag{5.28}$$

and where the $\pm\mathrm{i}0$ refer to just before and after the cut. Putting this all together, we find that the resolvent is given by

$$G(x) = \frac{1}{2x}\left(1 + \sqrt{(2\pi n x - 1)^2 + 8\pi m x}\right) - \pi n. \tag{5.29}$$

For large x, we see that $G(x)$ behaves as

$$G(x) \underset{x\to\infty}{\simeq} \frac{m/n}{x}, \tag{5.30}$$

and so the filling fraction is $\alpha = m/n$. Expanding $G(x)$ about zero, we also see that

$$E = -\frac{\lambda}{8\pi^2 L} G'(0) = \frac{\lambda m(n - m)}{2L}. \tag{5.31}$$

6. Comparison to circular solutions in string theory

In this section we will compare our results in the previous section to results from string theory. Over the last few years there has been much progress in this area, with surprising agreement between the string predictions and the gauge theory predictions coming from their spin chain structure.

The AdS/CFT correspondence is a conjectured duality between $\mathcal{N} = 4$ SYM and type IIB string theory on the ten-dimensional background $\mathrm{AdS}_5 \times S^5$. For our purposes the duality means that for every gauge invariant operator there is a corresponding string state, and the dimension of the operator is equal to the energy of the string in appropriate units. The radius R of the AdS_5 and S^5 are equal and has the relation to the gauge coupling, $R^4 = \lambda\alpha'^2$, where α' is inversely related to the string tension. We also note that the isometry group of this space is $SO(2,4) \times SO(6)$, which corresponds to the conformal symmetry and the internal R-symmetry of the gauge theory.

Unfortunately, solving string theory in this background is a difficult, and so far, unsolved problem. The main stumbling block is that not only is the space-time curved, but there is a background flux which comes from the Ramond–Ramond sector. There has yet to be found a practicable method for dealing with such fluxes. Nevertheless, string theory in $\mathrm{AdS}_5 \times S^5$ is known to be classically integrable [18], so while we have yet to find a full quantum treatment of the theory, we can still consider its classical behaviour and make comparisons to the corresponding classical behaviour in the gauge theory. We will restrict our discussion to the classical solutions that correspond to the one-cut solution of the previous section.

Since we are staying within the $SU(2)$ sector, this means that on the string theory side the strings are constrained to move on the subspace $R_t \times S^3$, where R_t is the global time component of the AdS_5 and the S^3 is a subspace of S^5. The isometry group of S^3 is $SO(4) \simeq SU(2) \times SU(2)$, so there is an extra $SU(2)$ symmetry. It is convenient to parameterize the S^3 subspace with the two by two matrix

$$g \equiv \begin{pmatrix} Z & W \\ -\overline{W} & \overline{Z} \end{pmatrix} \qquad \det g = 1. \tag{6.1}$$

The $SU(2) \times SU(2)$ isometry group then corresponds to left and right $SU(2)$ group multiplication on g. From this point of view, we see that the $SU(2)_L$ doublets, corresponding to left multiplication, are

$$\begin{pmatrix} Z \\ -\overline{W} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} W \\ \overline{Z} \end{pmatrix}, \tag{6.2}$$

while the $SU(2)_R$ doublets , corresponding to right multiplication, are

$$(Z \quad W) \qquad \text{and} \qquad (-\overline{W} \quad \overline{Z}). \tag{6.3}$$

We see that Z and W are both top components of $SU(2)_L$, but Z is a top component of $SU(2)_R$ and W is a bottom component. Hence, $SU(2)_R$ corresponds to the $SU(2)$ sector we have chosen for the gauge theory.

To find the solutions for the string motion, we consider the polyakov action

$$S = \frac{1}{4\pi\alpha'} \int \mathrm{d}\tau\, \mathrm{d}\sigma \sqrt{-h} h^{\alpha\beta} G_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu, \tag{6.4}$$

where $G_{\mu\nu}$ ($h_{\alpha\beta}$) is the metric on the target space (world-sheet). In conformal gauge we can choose $h_{00} = -1$, $h_{11} = 1$ and $t = \kappa\tau$. The target space metric is

$$\mathrm{d}s^2 = R^2(-\mathrm{d}t^2 + \mathrm{d}Z\, \mathrm{d}\overline{Z} + \mathrm{d}W\, \mathrm{d}\overline{W}), \tag{6.5}$$

with the constraint $Z\overline{Z} + W\overline{W} = 1$. The action then becomes

$$S = \frac{\sqrt{\lambda}}{4\pi} \int \mathrm{d}\tau\, \mathrm{d}\sigma (-\kappa^2 + \dot{Z}\dot{\overline{Z}} + \dot{W}\dot{\overline{W}} - Z'\overline{Z}' - W'\overline{W}' + \Lambda(Z\overline{Z} + W\overline{W} - 1)), \tag{6.6}$$

where Λ is a Lagrange multiplier to enforce the constraint. The equations of motion for Z and W are

$$\ddot{Z} - Z'' + \Lambda Z = 0 \qquad \ddot{W} - W'' + \Lambda W = 0 \tag{6.7}$$

and the solution for Λ is

$$\Lambda = \dot{Z}\dot{\overline{Z}} + \dot{W}\dot{\overline{W}} - Z'\overline{Z}' - W'\overline{W}'. \tag{6.8}$$

Varying S with respect to $h_{\alpha\beta}$ leads to the Virasoro constraints

$$\kappa^2 = \dot{Z}\dot{\overline{Z}} + \dot{W}\dot{\overline{W}} + Z'\overline{Z}' + W'\overline{W}' \qquad 0 = \dot{Z}\overline{Z}' + Z'\dot{\overline{Z}} + \dot{W}\overline{W}' + W'\dot{\overline{W}}. \tag{6.9}$$

There are three isometries of the metric with corresponding conserved quantities. The isometries are translations along the t direction and the phases in Z and W. The conserved quantities are the energy

$$E = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} \mathrm{d}\sigma\,\dot{t} = \sqrt{\lambda}\kappa, \tag{6.10}$$

and the two angular momenta

$$J_1 = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} \mathrm{d}\sigma(-\mathrm{i}\overline{Z}\dot{Z}) \qquad J_2 = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} \mathrm{d}\sigma(-\mathrm{i}\overline{W}\dot{W}). \tag{6.11}$$

We would expect J_1 and J_2 to count the number of Z and W fields in the gauge theory operator.

If we now make the following ansatz for the solutions [19, 20]:

$$Z = \cos\theta_0 \exp(\mathrm{i}(\omega_1\tau + m_1\sigma)), \qquad W = \sin\theta_0 \exp(\mathrm{i}(\omega_2\tau + m_2\sigma)), \tag{6.12}$$

then the equations of motion are satisfied if

$$\omega_1^2 - m_1^2 = \omega_2^2 - m_2^2. \tag{6.13}$$

We also have that the angular momenta are

$$J_1 = \sqrt{\lambda}\omega_1 \cos^2\theta_0, \qquad J_2 = \sqrt{\lambda}\omega_2 \sin^2\theta_0, \tag{6.14}$$

and the Virasoro constraints are

$$\begin{aligned} E^2 &= \frac{J_1^2}{\cos^2\theta_0} + \lambda m_1^2 \cos^2\theta_0 + \frac{J_2^2}{\sin^2\theta_0} + \lambda m_2^2 \sin^2\theta_0 \\ 0 &= m_1 J_1 + m_2 J_2. \end{aligned} \tag{6.15}$$

Using these equations, we can express the energy in terms of three quantities, say m_1, m_2 and J_1. In general, the result can be written as an expansion in λ, with the first two terms given by [20]

$$E = L - \frac{\lambda m_1 m_2}{2L} + \cdots, \tag{6.16}$$

where $L = J_1 + J_2$. Note that the first term in the energy corresponds to the bare dimension of the operator. The second term should correspond to the one-loop correction. This expression is consistent with the one-cut solution of the previous section if we identify $m_1 = m$ and $m_2 = -(n - m)$. If $m_1 = -m_2 \equiv m$, then and one finds that $J_1 = J_2$ and the energy is simply [19]

$$E = \sqrt{L^2 + \lambda m^2}. \tag{6.17}$$

We should point out that the classical string theory computation will be reliable if the quantum corrections are small. If we examine (6.6), we see that $1/\sqrt{\lambda}$ plays the role of $\hbar$, and so the string computations are reliable if $\lambda \gg 1$. However, perturbation theory in the gauge theory is best under control if $\lambda \ll 1$. Hence, it seems strange to be comparing the string and gauge solutions. The reason that we are able to make the comparison is because the effective expansion parameter is actually λ/L^2. This is known as BMN scaling [11]. Hence, this can be made small on the string side while still keeping λ large. On the gauge theory side, if there is truly BMN scaling then λ can be made large and still perturbation theory is under control, allowing for a direct comparison between the string and the gauge theory.

If one continues these arguments to two-loop order, one still finds agreement between the gauge theory and the string theory results [17, 21], but at three-loop order the results begin to differ [21]. These differences should not be taken as a refutation of the AdS/CFT correspondence. Instead this is most likely caused by a breakdown of BMN scaling on the gauge theory side.

7. Suggestions for further reading

In this final section we give a brief list of references for further reading.

The most comprehensive review of the AdS/CFT correspondence is in [22]. Among other things, there is a very good exposition of the superconformal algebra. There is also a comprehensive list of references to other works. This was written in 1999 and of course much has happened since then.

The major breakthrough after the original flurry of papers in AdS/CFT was the paper by Berenstein, Maldacena and Nastase [11]. They showed how to go beyond the protected operators in the comparison of gauge and string theory predictions. In the spin chain language, they are considering long chains with small numbers of magnons and small momentum. In this limit the interactions between the magnons can be ignored. In [23] the $1/L$ corrections on the string side were computed.

Integrability in gauge theory was first seen by Lipatov in high energy QCD [24]. Essentially QCD becomes conformal at high energies and the anomalous dimensions of operators with covariant derivatives can be computed. Soon thereafter, Faddeev and Korchemsky mapped this problem to a spin chain with spins in the fundamental representation of $SL(2)$ [25]. There has also been a lot of follow up work to this approach [26].

The original work on integrability in $\mathcal{N} = 4$ was in [1] for the scalar operators. The one-loop generalization to the full set of operators was carried out in [5]. Higher loop integrability was first investigated by Beisert, Kristjansen and Staudacher [6], where it was shown that the $SU(2)$ sector is integrable to the first 2 orders in perturbation theory. Assuming that the theory remains integrable to higher orders, these authors were able to derive the dilatation operator to three-loop order. This was later explicitly confirmed by actual field theory calculations [27–29].

Semiclassical analysis of string motion on $\mathrm{AdS}_5 \times S^5$ was first considered in [30]. This was later followed up by many other works [12, 19, 20, 31–36]. The classical integrability of string theory on $\mathrm{AdS}_5 \times S^5$ was first shown for the bosonic part in [37] and the full theory in [18]. Its relation to a Yangian symmetry was described in [38]. In [17], the classical integrability was exploited for string theory on $R_t \times S^3$ to show that all classical string solutions of the 'finite gap' type were consistent with the gauge theory predictions to second order in λ. Previous to this, Serban and Staudacher showed that there was consistency for certain examples, but they also showed that the third order results differed from each other [21]. This was also apparent in the work of [23], where the $1/L$ corrections to BMN on the string theory side differed from results obtained in [6]. In [39], some difficulties in even comparing the gauge and string results were discussed. Generalizations of [17] to other sectors have also been determined [40–44].

Soon after [17], Beisert, Dippel and Staudacher proposed a Bethe ansatz for the full $SU(2)$ sector on the gauge side [45], and which was similar in structure to the integral equations in [17]. The ansatz is asymptotic only, which means that it gives the correct perturbative contribution only up to order λ^L for an operator of length L. Recently it was shown how to derive these same Bethe equations from the Hubbard model [46].

Using the integral equations in [17], Arutyunov, Frolov and Staudacher proposed a Bethe ansatz for the $SU(2)_R$ of string theory on $R_t \times S^3$ [47]. The Bethe equations necessarily differ from the BDS proposal, with an extra phase factor in the S-matrix. The AFS proposal recently received some confirmation by Gromov and Kazakov, who showed that the AFS proposal was consistent with strings quantized on $R_t \times S^3$ [48].

Another approach to studying the AdS/CFT string correspondence is to study an effective action for the Yang–Mills sector. This was first done by Kruczenski [49], who showed that the classical limit of the XXX model, the Landau–Lifshitz model, can be extracted from the string sigma model in the limit of large L and leading order in λ. This was extended to other sectors and to higher loops in many other papers [50–62].

Quantum corrections to the classical string motion were first considered by Frolov and Tseytlin for circular strings on $R_t \times S^5$ [63]. Further progress was made in [64–69]. Hernandez and Lopez computed the one-loop corrections to the AFS phase factor [70]. Freyhult and Kristjansen then showed that the Hernandez–Lopez result is consistent with the one-loop corrections in the $SU(3)$ sector with two equal R-charges [71].

Much of the recent work has been focused on finding the complete S-matrix. In [72], Staudacher showed that it might be possible to find the S-matrix for the gauge or the string theory without actually knowing the dilatation operator. This was later extended in [73]. In [74], Beisert showed that the dispersion relation for magnon excitations are completely determined by the assumption of integrability and the superconformal algebra. Even more surprising, the S-matrix is also completely determined, up to an overall phase. In [75], Janik argued that this phase factor would satisfy a single equation if the spin chain has a symmetry analogous to crossing symmetry in a relativistic theory. It was quickly shown that the AFS phase factor is consistent with this equation to leading order [76]. In [77], a solution was given to the Janik equation up to a homogeneous term that hopefully can be eventually determined by matching it to the classical string behaviour.

There has also been some interesting work on finding the quantum version of the string theory. In [78] Mann and Polchinski start with a known integrable conformal model, the $OSp(2m+2|2m)$ supercoset. This theory has an $SU(2)$ sector and they are able to reproduce the classical results in [17] with a large number of excitations. A similar study was done in [79], where they consider the sigma model on $R \times S^n$. This theory is not conformal, it is actually asymptotically free, but nevertheless they are still able to get the classical results in [17]. In [80], Klose and Zarembo study the quantization of the Landau–Lifshitz model as well as the sigma model of $R \times S^3$ with Virasoro constraints.

Finally, other reviews on integrability in $\mathcal{N} = 4$ are Beisert's thesis [8], as well as reviews by Tseytlin [81], Zarembo [82] and Plefka [83]. An excellent review of the Bethe ansatz, including its applications to other systems as well as a thorough discussion of the S-matrix and its analyticity properties can be found in the review by Faddeev [84].

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